Journal of Number Theory 132 (2012) 348–370

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Non-cyclic class groups and the Brumer–Stark conjecture

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ARTICLE INFO

Article history:
Received 6 November 2008
Revised 7 May 2011
Accepted 21 May 2011
Communicated by David Goss

Keywords:
Algebraic number field
Artin L-function
Brumer–Stark conjecture

ABSTRACT

For an odd prime number p, we consider the p-primary part of the Brumer–Stark conjecture for a cyclic extension K/k of number fields of degree $2p$. We extend earlier work of Greither, Roblot, and Tangedal (2004) [4] by proving the conjecture when the minus component of the p-primary part of the class group of K is not a cyclic Galois module. Consequently, we are able to prove the full Brumer–Stark conjecture for some new classes of number field extensions.

1. Introduction

Let K/k be an Abelian extension of number fields with Galois group G. Let S be a finite set of places of k containing the Archimedean places and the prime ideals that ramify in K/k. To avoid trivial complications, we assume that k is totally real and K is totally complex (so that $|S| \geqslant 2$). If $\mathfrak{p}$ is a prime ideal of k not contained in S, we denote the Frobenius automorphism in G associated with $\mathfrak{p}$ by $\sigma_{\mathfrak{p}}$. For each χ in the group $\hat{G}$ of complex-valued characters of G, the Abelian L-function deprived of the Euler factors corresponding to primes in S is defined by

$$L_{K/k,S}(s,\chi) = \prod_{\mathfrak{p}\notin S}\left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{N\mathfrak{p}^s}\right)^{-1}.$$

This converges absolutely and uniformly on compact subsets of $\Re(s) > 1$, and each function $L_{K/k,S}(s,\chi)$ has an analytic continuation to an entire function excepting a simple pole at $s=0$ when χ is the trivial character. The L-function evaluator $\theta = \theta_{K/k,S}$ is the element of the complex group algebra $\mathbb{C}[G]$ defined by

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2011.05.016

$$\theta = \sum_{\chi \in \hat{G}} L_{K/k,S}(0, \overline{\chi}) e_\chi,$$

where e_χ is the idempotent corresponding to χ.

Let μ_K be the group of roots of unity in K, and let $\mathrm{Ann}_{\mathbb{Z}[G]}\,\mu_K$ be its $\mathbb{Z}[G]$-annihilator. The L-function evaluator has the following properties:

Rationality Property

$$\theta \in \mathbb{Q}[G].$$

Integrality Property

$$\mathrm{Ann}_{\mathbb{Z}[G]}\,\mu_K \cdot \theta \subseteq \mathbb{Z}[G].$$

The rationality property was first proved by Siegel [12], building on earlier work of Klingen [6]. It was reproved by Shintani [11], who gave explicit formulae for the coefficients of θ. The integrality property was proved independently first by Deligne and Ribet [3], building on Siegel's work, and then later by Cassou-Noguès [2] and Barsky [1], who extended Shintani's methods.

The rationality property and the definition of θ immediately imply the following result:

Theorem 1.1. *If χ is a character of order n, then*

$$L_{K/k,S}(0, \chi) \in \mathbb{Q}(\zeta_n).$$

Let $W = |\mu_K|$. The integrality property implies the following strengthened rationality property:

$$\theta \in \frac{1}{W}\mathbb{Z}[G].$$

We call an element x of $K^\times$ an anti-unit if $|\phi(x)| = 1$ for each embedding $\phi : K \hookrightarrow \mathbb{C}$. If n is a divisor of W, we call an element x in K n-Abelian (for K/k) if the extension $K(\sqrt[n]{x})/k$ is Abelian.

Definition 1.2. Suppose that α is in $\mathbb{Z}[G]$ and $\mathfrak{a}$ is a fractional ideal of K. We say that α is a BS_n-annihilator for $\mathfrak{a}$ if there exists an element $\varepsilon(\mathfrak{a}, K/k, S)$ in K such that

(1) $\mathfrak{a}^\alpha = (\varepsilon(\mathfrak{a}, K/k, S))$.
(2) $\varepsilon(\mathfrak{a}, K/k, S)$ is an anti-unit.
(3) $\varepsilon(\mathfrak{a}, K/k, S)$ is n-Abelian.

Conditions (1) and (2) determine the element $\varepsilon(\mathfrak{a}, K/k, S)$ up to a factor of a Wth root of unity (see [16, Lemma 1.6]), and condition (3) is unaffected by this factor. If $\mathfrak{S}$ is a collection of fractional ideals of K, we say that α is a BS_n-annihilator for $\mathfrak{S}$ if it is a BS_n-annihilator for every fractional ideal in $\mathfrak{S}$. When $n = W$, we say simply that α is a BS-annihilator for $\mathfrak{S}$. Note that if α is a BS_n-annihilator for a set of fractional ideals $\mathfrak{S}$ of K, then it is a BS_n-annihilator for the group of fractional ideals generated by $\mathfrak{S}$. Note further that the set of BS_n-annihilators for $\mathfrak{S}$ is an ideal in $\mathbb{Z}[G]$.

The Brumer–Stark conjecture for K/k and S generalizes the analytic class number formula:

The Brumer–Stark conjecture. *$W\theta$ is a* BS*-annihilator for the group of nonzero fractional ideals of K.*

Let p be a prime number, and assume that the number of p-power roots of unity in K is p^r. The conjecture has a local formulation:

The *p*-primary Brumer–Stark conjecture. *$W\theta$ is a BS_{p^r}-annihilator for the fractional ideals representing classes in the p-primary part $\mathrm{Cl}_K\{p\}$ of Cl_K.*

The Brumer–Stark conjecture is equivalent to the collection of its p-primary versions for all prime numbers p [4, Proposition 1.1].

Classifying extensions by their Galois groups, the only extensions for which the Brumer–Stark conjecture is known to be true without exception are quadratic [15] and biquadratic [9]. Because of the difficulty of proving the conjecture for extensions with other Galois groups, much effort has been directed instead toward proving the p-primary conjecture for various primes p. The state of the art is the following recently announced result of Greither and Popescu: if

- the p-cyclotomic Iwasawa μ-invariant of K is 0, and
- p is an odd prime

then the p-primary Brumer–Stark conjecture is true if p does not divide the degree of the extension, and an imprimitive version is true if p does divide the degree of the extension. More precisely, if p divides the degree of the extension, then they require the set S to contain the prime ideals of k dividing p, so S need not be the minimal possible set.

In this paper, we study the simplest case where Greither and Popescu obtained only an imprimitive result: the p-primary Brumer–Stark conjecture for extensions of degree $2p$, where p is an odd prime. Our method uses no Iwasawa theory, so makes no assumption regarding the μ-invariant or the set S. Our method also indicates the amount of weakening caused by adding primes to S. In fact, all cases not provable by our method become accessible to our method if a single place with Frobenius of order p is added to the set S. It appears then that a substantially deeper method is required to prove the conjecture for certain very special extensions when using the minimal set S, and it is the author's hope that the present paper gives a better understanding of exactly which extensions these are.

We make the following convention: K_1/k_0 will be used to denote an Abelian degree $2p$ extension of number fields, and K/k will be used for other extensions.

In [4], the authors made progress studying cyclic extensions of degree $2p$ for odd primes p, focusing especially on the 2- and p-primary local conjectures. Let K_1/k_0 be a cyclic extension of degree $2p$ with k_0 totally real and K_1 totally complex, and let K_0 be the intermediate field such that $[K_0 : k_0] = 2$. Let ζ_p denote a primitive pth root of unity. They proved that the p-primary Brumer–Stark conjecture holds outside of two cases, which they termed $\sharp$ and $\flat$:

$\sharp$: $\zeta_p \in K_1$ and no prime of k_0 splits in K_0 and ramifies in K_1.
$\flat$: $\zeta_p \notin K_1$, no prime of k_0 splits in K_0 and ramifies in K_1, and

$$K_1^{\mathrm{cl}} \subseteq \left(K_1^{\mathrm{cl}}\right)^+(\zeta_p).$$

Here, cl indicates the normal closure over $\mathbb{Q}$ and $+$ indicates the maximal totally real subfield.

Remark. In [4], the authors called the above class $\sharp$ of extensions "subcase II(b)" and erroneously equated this with a different class of extensions, which they called $\sharp$. In fact, their class $\sharp$ properly contains the class of "subcase II(b)" extensions, but this does not significantly affect the theoretical results in their paper.

The authors indicate that case $\sharp$ should be viewed as somewhat exceptional, whereas case $\flat$ should be viewed as even more exceptional.

The analysis of case $\sharp$ naturally breaks into two subcases:

Case $A^\sharp$: K_1/k_0 is in case $\sharp$ and K_1 is generated over k_0 by p-power roots of unity.
Case $B^\sharp$: K_1/k_0 is in case $\sharp$ and K_1 is not generated over k_0 by p-power roots of unity.

In this paper, we will prove the p-primary Brumer–Stark conjecture for cyclic extensions K_1/k_0 of degree $2p$ in case $\sharp$ or $\flat$ for which the minus component of the p-primary class group $\mathrm{Cl}_{K_1}^-\{p\}$ is not cyclic as a $\mathbb{Z}[G]$-module. It will be shown that for such extensions, this is equivalent to $\mathrm{Cl}_{K_0}^-\{p\}$ not being a cyclic group. We also show that if we allow S to be imprimitive, then the p-primary Brumer–Stark conjecture holds if S contains even one unramified place with inertia degree p. Combined with the results in [4], this will imply our main theorem:

Theorem 1.3. *Let K_1/k_0 be an extension of number fields with cyclic Galois group G of order $2p$, and let S be a set of places of k_0 containing the Archimedean places and the prime ideals that ramify in K_1/k_0. If $\mathrm{Cl}_K^-\{p\}$ is not a cyclic $\mathbb{Z}[G]$-module or if S contains an unramified prime ideal with inertia degree p, then the p-primary Brumer–Stark conjecture for K/k is true.*

In Section 2, we provide the fundamental link between θ and the arithmetic of K. In Section 3, we provide results about ranks of pieces of class groups which are used in the proof of Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.3. Finally, in Section 5, we will use Theorem 1.3 to prove the full Brumer–Stark conjecture for two classes of field extensions (Theorems 5.4 and 5.6).

2. Properties of θ for degree $2p$ extensions

We begin by fixing some notation that will be used throughout this paper. Let K_1/k_0 be an Abelian extension of degree $2p$ with Galois group G. We assume that k_0 is totally real and that K_1 is totally complex. Let k_1/k_0 and K_0/k_0 be the extensions of k_0 contained in K_1 of degrees p and 2 respectively; k_1 is totally real and K_0 and K_1 are CM fields. For $i = 0, 1$, let $W_i = |\mu_i|$ be the cardinality of the group of roots of unity in K_i. Let $\tilde{\mu}$ be the quotient μ_1/μ_0, and set

$$q = \frac{W_1}{W_0} = |\tilde{\mu}|.$$

We note that the exact power of p dividing q is either p^0 or p^1. Let τ and σ be elements of orders 2 and p in G (so that τ is complex conjugation), and set $H = \langle\sigma\rangle = \mathrm{Gal}(K_1/K_0)$. Let N_H be the algebraic norm in $\mathbb{Z}[G]$ associated with the subgroup H of G. We fix a generator χ of $\hat{G}$ and let ζ_p be the primitive pth root of unity such that $\chi(\sigma) = \zeta_p$.

If K is one of the fields mentioned above, let S_K be the set of places of K composed of the Archimedean places and the prime ideals dividing primes in k_0 that ramify in K_1/k_0. Let S_K^{ns} be the subset of S_K comprising the places dividing those in S_{k_0} that do not split in K_0/k_0. Finally, let S be a set of places of k_0 containing S_{k_0}. We assume that no place in S splits completely in K_1/k_0 since otherwise $\theta_{K_1/k_0,S} = 0$ and the Brumer–Stark conjecture is trivial. Let S_1 and S_2 be the subsets of S composed of the places that split in k_1/k_0 and the places that split in K_0/k_0 and are unramified in k_1/k_0 respectively.

Let $\mathfrak{C}_0$ and $\mathfrak{C}_1$ denote the cokernels of the canonical maps of S class groups $\mathrm{Cl}_{k_0,S_{k_0}^{\mathrm{ns}}} \to \mathrm{Cl}_{K_0,S_{K_0}^{\mathrm{ns}}}$ and $\mathrm{Cl}_{k_1,S_{k_1}^{\mathrm{ns}}} \to \mathrm{Cl}_{K_1,S_{K_1}^{\mathrm{ns}}}$. The groups $\mathfrak{C}_0$ and $\mathfrak{C}_1$ will appear frequently in what follows. To understand them better, we begin with a lemma showing that away from the 2-primary part, they are simply the minus parts of the class groups Cl_{K_0} and Cl_{K_1}.

Lemma 2.1. *If l is an odd prime, then there are isomorphisms*

$$(\mathrm{Cl}_{K_0} \otimes \mathbb{Z}_l)^{1-\tau} \to \mathfrak{C}_0 \otimes \mathbb{Z}_l,$$

$$(\mathrm{Cl}_{K_1} \otimes \mathbb{Z}_l)^{1-\tau} \to \mathfrak{C}_1 \otimes \mathbb{Z}_l.$$

Proof. Let $i = 0$ or 1. The surjection $\mathrm{Cl}_{K_i} \to \mathfrak{C}_i$ induces, after tensoring with $\mathbb{Z}_l$, a surjection

$$\mathrm{Cl}_{K_i} \otimes \mathbb{Z}_l \to \mathfrak{C}_i \otimes \mathbb{Z}_l.$$

If $\mathfrak{a}$ is a nonzero fractional ideal of K_i, then $\mathfrak{a}^{1+\tau}$ is a lift from k_i, so the image of $\mathfrak{a}^{1+\tau}$ in $\mathfrak{C}_i$ is trivial. It follows that restricting the above map to the minus part of $\mathrm{Cl}_{K_i} \otimes \mathbb{Z}_l$ gives a surjection

$$\phi : (\mathrm{Cl}_{K_i} \otimes \mathbb{Z}_l)^{1-\tau} \to \mathfrak{C}_i \otimes \mathbb{Z}_l.$$

To see that ϕ is injective, let $\mathfrak{a}^{1-\tau}$ be a nonzero fractional ideal representing a class in $(\mathrm{Cl}_{K_i} \otimes \mathbb{Z}_l)^{1-\tau}$ whose image in $\mathfrak{C}_i \otimes \mathbb{Z}_l$ is trivial. Then $\mathfrak{a}^{1-\tau} = \mathfrak{b}\mathfrak{c}(\gamma)$, where $\mathfrak{b}$ is an ideal divisible only by primes in $S^{\mathrm{ns}}_{K_i}$, $\mathfrak{c}$ is a lift from k_i, and γ is in $K_i^\times$. Since primes in $S^{\mathrm{ns}}_{K_i}$ are fixed by τ, we have

$$\mathfrak{a}^{2-2\tau} = \mathfrak{b}^{1-\tau}\mathfrak{c}^{1-\tau}\left(\gamma^{1-\tau}\right) = \left(\gamma^{1-\tau}\right).$$

Thus, $\mathfrak{a}^{1-\tau}$ is trivial in $\mathrm{Cl}_{K_i} \otimes \mathbb{Z}_l$ and ϕ is injective. □

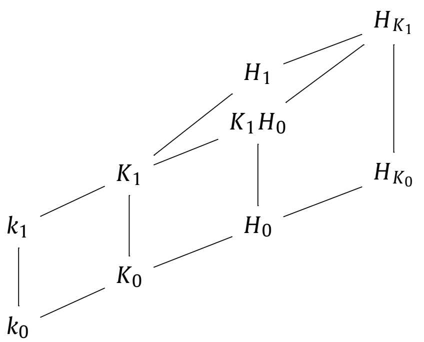
The following proposition is fundamental:

Proposition 2.2. *The norm map of ideals induces a* surjective *map*

$$N : \mathfrak{C}_1 \to \mathfrak{C}_0.$$

Proof. For $i = 0$ and 1, let H_{K_i} be the Hilbert class field of K_i, and let H_i be the subfield of H_{K_i} corresponding to $\mathfrak{C}_i$ through class field theory. Let $\tilde{\tau}$ be the restriction of complex conjugation to K_0. The extension H_{K_0}/k_0 is Galois, so $\tilde{\tau}$ acts on $\mathrm{Gal}(H_{K_0}/K_0)$ by the lift-and-conjugate action. The Artin map $\mathrm{Cl}_{K_0} \to \mathrm{Gal}(H_{K_0}/K_0)$ preserves the $\tilde{\tau}$-action. Thus, $\tilde{\tau}$ acts by inversion on $\mathrm{Gal}(H_0/K_0)$ since it acts by inversion on $\mathfrak{C}_0$. We cannot have $K_1 \subseteq H_0$ since $\tilde{\tau}$ acts trivially on $H = \mathrm{Gal}(K_1/K_0)$ and $|H|$ is an odd prime. Thus, $K_1 \cap H_0 = K_0$.

We observe that K_1H_0 is Abelian over K_0 and unramified over K_1. Thus, we have the following diagram of fields:



Let $\mathfrak{a}$ be an ideal of K_1 whose class in $\mathfrak{C}_1$ is trivial. We may write $\mathfrak{a} = \mathfrak{b}\mathfrak{c}(\gamma)$, where $\mathfrak{b}$ is an ideal of K_1 supported at prime ideals in $S^{\mathrm{ns}}_{K_1}$, $\mathfrak{c}$ is the lift of an ideal from k_1, and γ is in $K_1^\times$. If N denotes the norm map from K_1 to K_0, then

$$N\mathfrak{a} = N\mathfrak{b}N\mathfrak{c}(N\gamma).$$

Here, $N\mathfrak{b}$ is supported at primes in $S^{\mathrm{ns}}_{K_0}$, and $N\mathfrak{c}$ is the lift of an ideal from k_0 to K_0. Therefore, the class of $N\mathfrak{a}$ in $\mathfrak{C}_0$ is trivial, and the norm map of ideals induces a map $N : \mathfrak{C}_1 \to \mathfrak{C}_0$. We must show that it is surjective.

Letting $(\cdot, K/k)$ denote the Artin symbol for an Abelian extension K/k, it follows from standard properties of the Artin symbol [7, p. 198] that

$$(\mathfrak{a}, H_{K_1}/K_1)|_{H_0} = (N\mathfrak{a}, H_0/K_0) = (N\mathfrak{a}, H_{K_0}/K_0)|_{H_0} = \mathrm{id}$$

since H_0 is the subfield of H_{K_0} fixed by the automorphisms $(\mathfrak{f}, H_{K_0}/K_0)$ for ideals $\mathfrak{f}$ in K_0 representing the trivial class in $\mathfrak{C}_0$. This equality then shows that $K_1H_0 \subset H_1$. There is thus a restriction map $\mathrm{Gal}(H_1/K_1) \to \mathrm{Gal}(H_0/K_0)$, which is surjective since $K_1 \cap H_0 = K_0$. It follows that the corresponding norm map $N : \mathfrak{C}_1 \to \mathfrak{C}_0$ is surjective. $\square$

We denote the kernel of the norm map in Proposition 2.2 by $\mathfrak{K}$ and its p-primary part by $\mathfrak{K}_p$. We also let A_0 and A_1 be the p-primary parts of the $\mathbb{Z}[G]$-modules $\mathfrak{C}_0$ and $\mathfrak{C}_1$.

Lemma 2.3. *If no prime ideal of k_0 splits in K_0/k_0 and ramifies in k_1/k_0, then*

$$\mathfrak{K}_p = A_1^{1-\sigma}.$$

Proof. Lemma 2.1 shows that A_0 and A_1 are isomorphic (as G-modules) to the minus components of the p-primary parts of Cl_{K_0} and Cl_{K_1}. Lemma 2.5 in [4] states that

$$\left|A_1^H\right| = |A_0| p^t,$$

where t is the number of primes in k_0 that split in K_0/k_0 and ramify in k_1/k_0. By our assumption, $t = 0$. Therefore,

$$|\mathfrak{K}_p| = \frac{|A_1|}{|A_0|} = \frac{|A_1|}{|A_1^H|} = \left|A_1^{1-\sigma}\right|.$$

But $A_1^{1-\sigma} \subseteq \mathfrak{K}_p$ by the definition of $\mathfrak{K}$, so in fact $\mathfrak{K}_p = A_1^{1-\sigma}$. $\square$

We now begin our analysis of $\theta = \theta_{K_1/k_0,S}$ by splitting it into two pieces:

$$\theta_0 = L_{K_1/k_0,S}\big(0, \chi^p\big) e_{\chi^p}$$

and

$$\theta_1 = \sum_{\substack{i=1 \\ i \text{ odd} \\ i \neq p}}^{2p-1} L_{K_1/k_0,S}\big(0, \overline{\chi}^i\big) e_{\chi^i}.$$

Since $L_{K_1/k_0,S}(0, \chi^i) = 0$ when i is even (including when $i = 0$ since $|S| \geqslant 2$), $\theta = \theta_1 + \theta_0$. Theorem 1.1 immediately implies that θ_0 is in $\mathbb{Q}[G]$; it follows that θ_1 is in $\mathbb{Q}[G]$ as well.

The arithmetic properties of θ_0 and θ_1 we will need arise from the relative analytic class number formula for quadratic extensions. Tate [15, section 3, (c)] provided the following explicit expression for the L-function evaluator for a CM extension K/k and set S when no prime ideal in S splits in K/k:

$$\theta_{K/k,S} = \frac{2^{|S|-2}|\mathfrak{C}|}{W}(1-\tau). \tag{1}$$

Here, τ is complex conjugation, W is the number of roots of unity in K, and $\mathfrak{C}$ is the cokernel of the canonical map of S-class groups $\mathrm{Cl}_{k,S} \to \mathrm{Cl}_{K,S}$.

The following two propositions give the important properties of θ_0 and θ_1.

Proposition 2.4. *The element θ_0 can be written as*

$$\theta_0 = \frac{N_H}{p}\tilde{\theta}_0,$$

where $\tilde{\theta}_0$ is any lift of $\theta_{K_0/k_0,S}$ to $\mathbb{Q}[G]$ (through the natural projection map). If a place in S splits in K_0, then $\theta_0 = 0$; otherwise, $\theta_0 \neq 0$.

Proof. Let $\pi : \mathbb{Z}[G] \to \mathbb{Z}[\mathrm{Gal}(K_0/k_0)]$ be the projection map. Using the definition of θ_0 and the inflation property of Artin L-functions, we find that θ_0 is a lift of $\theta_{K_0/k_0,S}$ through π. If β and β' are two lifts of $\theta_{K_0/k_0,S}$ through π, then $N_H(\beta - \beta') = 0$. Thus,

$$\frac{N_H}{p}\tilde{\theta}_0$$

does not depend on the choice of lift of $\theta_{K_0/k_0,S}$. In particular, we may choose $\tilde{\theta}_0 = \theta_0$, and the given description of θ_0 follows. If a place of S splits in K_0/k_0, then $\theta_{K_0/k_0,S} = 0$ [15, section 1, Property 3]. Otherwise, $\theta_{K_0/k_0,S}$ is given by Tate's formula (1). $\square$

Proposition 2.5. *Writing $\theta = \theta_1 + \theta_0$ as above, the element θ_1 satisfies*

$$\begin{aligned} N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\big(\chi(\theta_1)\big) &= N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\big(L_{K_1/k_0,S}(0,\overline{\chi})\big) \\ &= \frac{1}{q}2^{(p-1)|S_1|}p^{|S_2|}|\mathfrak{K}|. \end{aligned}$$

Proof. The first equality follows from the orthogonality relations and the definition of θ_1. Next, we observe that the standard properties (additivity, inflation, induction) of Artin L-functions hold for S-imprimitive functions, with the convention that for a tower of fields $k \subset K' \subset K$, the induction property

$$L_{K/K',S'}(s,\psi) = L_{K/k,S}\big(s,\mathrm{Ind}(\psi)\big)$$

holds if the set S' is the set of places of K' dividing those in S. They follow from the corresponding properties of Artin L-functions and the fact that similar properties also hold for the individual Euler factors corresponding to the prime ideals in S. Let us use the nontrivial character ψ_1 on $\mathrm{Gal}(K_1/k_1)$ to induce a character on $\mathrm{Gal}(K_1/k_0)$. By Frobenius reciprocity, we find that the induced character is given by

$$\mathrm{Ind}(\psi_1) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{2p-1} \chi^i.$$

Letting ψ_0 denote the nontrivial character on $\mathrm{Gal}(K_0/k_0)$, it follows that

$$L_{K_1/k_1,S_{k_1}^{\mathrm{ns}}}(0,\psi_1) = \prod_{\substack{i=1 \\ i \text{ odd}}}^{2p-1} L_{K_1/k_0,S_{k_0}^{\mathrm{ns}}}\big(0,\chi^i\big) = L_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}(0,\psi_0)\prod_{\substack{i=1 \\ i \text{ odd} \\ i\neq p}}^{2p-1} L_{K_1/k_0,S_{k_0}^{\mathrm{ns}}}\big(0,\chi^i\big).$$

We would like to apply Theorem 1.1 to the L-function values in the product on the right, but for this, we need to remove Euler factors corresponding to prime ideals that ramify in k_1/k_0 and split in K_0/k_0. If $\mathfrak{p}$ is such a prime, then the corresponding Euler factor in $L_{K_1/k_0}(0, \chi^i)$ is

$$\frac{1}{\det(1-\sigma_{\mathfrak{p}}; V^{I_{\mathfrak{p}}})},$$

where $\sigma_{\mathfrak{p}}$ is an arbitrarily chosen Frobenius automorphism associated with $\mathfrak{p}$, V is the 1-dimensional representation corresponding to χ^i, and $V^{I_{\mathfrak{p}}}$ is the subspace of V fixed by the inertia group of $\mathfrak{p}$. Since $\mathfrak{p}$ splits in K_0/k_0 and ramifies in k_1/k_0, $I_{\mathfrak{p}} = H$. Since χ^i is nontrivial on H for $1 \leqslant i \leqslant 2p-1$ and $i \neq p$, the Euler factor is trivial. Thus, each L-function value $L_{K_1/k_0,S_{k_0}^{\mathrm{ns}}}(0, \chi^i)$ in the above product is equal to the L-function value $L_{K_1/k_0,S_{k_0}}(0, \chi^i)$.

Applying Theorem 1.1 and the rationality property of θ, the L-function values in the above product form a complete set of $\mathbb{Q}$-conjugate elements of $\mathbb{Q}(\zeta_p)$, so we obtain

$$\frac{L_{K_1/k_1,S_{k_1}^{\mathrm{ns}}}(0, \psi_1)}{L_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}(0, \psi_0)} = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\bigl(L_{K_1/k_0,S_{k_0}}(0, \overline{\chi})\bigr).$$

As no place in $S_{k_i}^{\mathrm{ns}}$ splits in K_i/k_i for $i = 0, 1$, we may apply ψ_1 to Tate's expression for $\theta_{K_1/k_1,S_{k_1}^{\mathrm{ns}}}$ (Eq. (1)) and ψ_0 to the expression for $\theta_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}$. This gives formulae for the L-functions values on the left side. Upon substitution, we find that

$$\begin{aligned} N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\bigl(L_{K_1/k_0,S_{k_0}}(0, \overline{\chi})\bigr) &= \frac{W_0}{W_1} 2^{|S_{k_1}^{\mathrm{ns}}| - |S_{k_0}^{\mathrm{ns}}|} \frac{|\mathfrak{C}_1|}{|\mathfrak{C}_0|} \\ &= \frac{1}{q} 2^{(p-1)|S^{\mathrm{spl}}|} |\mathfrak{K}|, \end{aligned}$$

where S^{spl} is the set of places in $S_{k_0}^{\mathrm{ns}}$ that split in k_1/k_0. In the second equality, we have used Proposition 2.2.

To obtain the second equality in the statement of the present proposition, we must remove the Euler factors corresponding to prime ideals in $S \setminus S_{k_0}$ from the L-function on the left. These primes are unramified in K_1/k_0, and the corresponding Euler factors have the form

$$\frac{1}{\det(1-\sigma_{\mathfrak{p}}; V)} = \frac{1}{1-\overline{\chi}(\sigma_{\mathfrak{p}})}.$$

Thus, we must multiply the L-function by the product of the factors $1 - \overline{\chi}(\sigma_{\mathfrak{p}})$ as $\mathfrak{p}$ runs through the prime ideals in S that are unramified in K_1/k_0. By assumption, none of these ideals splits completely in K_1/k_0. If $\mathfrak{p}$ is such an ideal, then

$$N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\bigl(1-\overline{\chi}(\sigma_{\mathfrak{p}})\bigr) = \begin{cases} 1, & \text{if } \mathfrak{p} \text{ is inert in } K_1/k_0, \\ 2^{p-1}, & \text{if } \mathfrak{p} \text{ splits in } k_1/k_0 \text{ and is inert in } K_0/k_0, \\ p, & \text{if } \mathfrak{p} \text{ splits in } K_0/k_0 \text{ and is inert in } k_1/k_0. \end{cases}$$

The formula for $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(L_{K_1/k_0,S}(0, \overline{\chi}))$ follows. $\square$

Lemma 2.6. *In $\mathbb{Z}[H]$, we have the identity*

$$\prod_{i=1}^{p-1}\left(1-\sigma^i\right) = p - N_H.$$

Proof. The $\mathbb{Z}$-linear extension of χ induces a ring isomorphism

$$\chi : \mathbb{Z}[H]/N_H \to \mathbb{Z}[\zeta_p].$$

Under this isomorphism, the above product has the same image as p. Therefore, there exists α in $\mathbb{Z}[G]$ such that

$$\prod_{i=1}^{p-1}\left(1-\sigma^i\right) = p + \alpha N_H = p + aN_H,$$

where a is the sum of the coefficients of α. Multiplying through by N_H, we find $0 = pN_H + apN_H$, so that $a = -1$ and the lemma follows. $\square$

Next, we will provide a bound on the denominators of the coefficients of θ_1.

Proposition 2.7.

$$\theta_1 \in \frac{1}{pq}\mathbb{Z}[G].$$

Proof. By the orthogonality relations, $N_H e_\psi = 0$ for each character $\psi \in \hat{G}$ which is nontrivial on H. Therefore, $N_H\theta_1 = 0$.

Next, $q(1-\sigma)$ is in $\operatorname{Ann}_{\mathbb{Z}[G]}\mu_1$, so the integrality property of θ and the formula for θ_0 in Proposition 2.4 show that

$$q(1-\sigma)\theta = q(1-\sigma)\theta_1 \in \mathbb{Z}[G].$$

Hence,

$$q\left(\prod_{i=1}^{p-1}\left(1-\sigma^i\right)\right)\theta_1 \in \mathbb{Z}[G].$$

From Lemma 2.6 and the fact that $N_H\theta_1 = 0$, it follows that $pq\theta_1$ is in $\mathbb{Z}[G]$. $\square$

Lemma 2.8. *If ω is in $\mathbb{Q}[G]$, then there exists $\alpha_\omega = \sum_{i=0}^{p-2} a_i\sigma^i$ in $\mathbb{Q}[H]$ such that*

$$\omega\theta_1 = (1-\sigma)\alpha_\omega(1-\tau).$$

If $\omega\theta_1 \in \mathbb{Z}[G]$, then α_ω can be chosen in $\mathbb{Z}[H]$.

Proof. Since $\chi^i(\theta_1) = 0$ when i is even, $\omega\theta_1$ has the form

$$\omega\theta_1 = \sum_{i=0}^{p-1} c_i \sigma^i (1-\tau).$$

Also, since $N_H\theta_1 = 0$, we have $\sum_{i=0}^{p-1} c_i = 0$. The lemma follows by setting

$$\alpha_\omega = \sum_{i=0}^{p-2}\left(\sum_{k=0}^{i} c_k\right)\sigma^i. \qquad \square$$

3. Results about ranks

In our analysis of the Brumer–Stark conjecture, we will consider the actions on Cl_{K_1} of integralized versions of the elements θ_0 and θ_1 from Section 2. The proof of the p-primary conjecture cannot be accomplished by analyzing these actions independently. Eq. (1) and Propositions 2.4 and 2.5 show that when both are nonzero, θ_0 is linked to $\mathfrak{C}_0$ and θ_1 is linked to $\mathfrak{K}$. In this section, we will gather the results relating the ranks of $\mathfrak{C}_0$ and $\mathfrak{K}$ that will be necessary for describing the interaction between θ_0 and θ_1.

If A is a finite Abelian group, we let ${}_pA$ be the subgroup of A annihilated by p and A_p be the p-primary part of A. We let $\mathrm{rk}_p(A)$ be the p-rank of A_p. If A is also an H-module, then A^H is the submodule fixed by H and ${}_{\mathrm{N}}A$ is the submodule of A annihilated by N_H.

We begin by analyzing the group $\mathfrak{C}_0$ defined in Section 2. As $S_{k_0}^{\mathrm{ns}}$ contains the prime ideals that ramify in K_0/k_0, $S_{k_0}^{\mathrm{ns}}$ is admissible as a set in the Brumer–Stark conjecture for the quadratic extension K_0/k_0. The following lemma is a refinement of the conjecture in this case.

Lemma 3.1. *Let $\tilde{p}$ be a prime number. If $r = \mathrm{rk}_{\tilde{p}}(\mathfrak{C}_0)$, then $\frac{1}{2^{|S_{k_0}^{\mathrm{ns}}|-2}\tilde{p}^{r-1}} W_0\theta_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}$ is a* BS*-annihilator for the group of nonzero fractional ideals in K_0.*

Under the assumption of the lemma, Tate's expression (1) shows that $2^{|S_{k_0}^{\mathrm{ns}}|-2}\tilde{p}^r$ divides the coefficients of $W_0\theta_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}$ and that $\frac{1}{2^{|S_{k_0}^{\mathrm{ns}}|-2}\tilde{p}^{r-1}} W_0\theta_{K_0/k_0,S_{k_0}^{\mathrm{ns}}}$ annihilates $\mathfrak{C}_0$. The proof of the lemma then follows in exactly the same way as Tate's proof of the Brumer–Stark conjecture for quadratic extensions (see [15, section 3, (c)]).

We now introduce the group $\mathfrak{K}$ into our analysis. Through the isomorphism $\chi : \mathbb{Z}[H]/N_H \to \mathbb{Z}[\zeta_p]$, $\mathfrak{K}$ is endowed with a $\mathbb{Z}[\zeta_p]$-module structure (which depends on the choice of χ). The $\mathbb{Z}[G]$-module action will be written with exponential notation and the $\mathbb{Z}[\zeta_p]$-module action with multiplicative notation.

The structure theorem for finitely generated torsion modules over Dedekind domains shows that the p-primary part of $\mathfrak{K}$ has a well-defined $(1-\zeta_p)$-rank, $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K})$. It is equal to the p-rank of each of the elementary p-groups $\mathfrak{K}^H$ and $\mathfrak{K}/\mathfrak{K}^{1-\sigma}$. The following lemmas demonstrate a connection between $\mathrm{rk}_p(\mathfrak{C}_0)$ and $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K})$.

We let

$$\iota : \mathfrak{C}_0 \to \mathfrak{C}_1$$

be the map induced by lifting fractional ideals.

Lemma 3.2. *If no prime ideal of k_0 splits in K_0/k_0 and ramifies in k_1/k_0, then*

$$\mathrm{rk}_p\big(\iota({}_p\mathfrak{C}_0)\big) \leqslant \mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) \leqslant \mathrm{rk}_p(\mathfrak{C}_0).$$

Proof. Consider the sequence of modified cohomology groups induced by the exact sequence

$$0 \to \mathfrak{K}_p \to A_1 \xrightarrow{\mathrm{N}} A_0 \to 0.$$

Using the definitions to simplify these groups (being careful to distinguish the norm *map* $\mathrm{N}: A_1 \to A_0$ from the norm *action* on A_1) and applying Lemma 2.3, we obtain

$$\cdots \to A_1^H / A_1^{N_H} \to A_0/A_0^p \to \mathfrak{K}_p/\mathfrak{K}_p^{1-\sigma} \xrightarrow{0} {}_{\mathrm{N}}\mathrm{A}_1/\mathfrak{K}_p \to {}_p\mathrm{A}_0 \to \mathfrak{K}_p^H \to \cdots.$$

Thus, the connecting homomorphism

$$A_0/A_0^p \to \mathfrak{K}_p/\mathfrak{K}_p^{1-\sigma}$$

is a surjection. The inequality

$$\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) \leqslant \mathrm{rk}_p(\mathfrak{C}_0)$$

follows. Also, the image of a class $\bar{\mathfrak{a}}$ in ${}_p\mathrm{A}_0$ under the penultimate arrow in the above long exact sequence is just the lift of $\bar{\mathfrak{a}}$ to A_1, giving the other inequality

$$\mathrm{rk}_p\big(\iota({}_p\mathfrak{C}_0)\big) \leqslant \mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}). \qquad \square$$

Lemma 3.3. *If no prime ideal of k_0 splits in K_0/k_0 and ramifies in k_1/k_0, then*

$$\mathrm{rk}_p(\mathfrak{C}_0) - 1 \leqslant \mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) \leqslant \mathrm{rk}_p(\mathfrak{C}_0).$$

In case $A^\sharp$ or case $\flat$,

$$\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) = \mathrm{rk}_p(\mathfrak{C}_0).$$

Proof. After Lemma 3.2, it only remains to show that,

$$\mathrm{rk}_p(\mathfrak{C}_0) - 1 \leqslant \mathrm{rk}_p\big(\iota({}_p\mathfrak{C}_0)\big), \tag{2a}$$

and that furthermore,

$$\mathrm{rk}_p(\mathfrak{C}_0) = \mathrm{rk}_p\big(\iota({}_p\mathfrak{C}_0)\big) \tag{2b}$$

in case $A^\sharp$ or case $\flat$. We may assume that p divides $|\mathfrak{C}_0|$, since these results are immediate otherwise.

Let $\mathfrak{a}$ be a fractional ideal of K_0 representing a class in A_0 of order p. We may write

$$\mathfrak{a}^p = \mathfrak{b}\mathfrak{c}(\gamma_{\mathfrak{a}}),$$

where $\gamma_{\mathfrak{a}}$ is in $K_0^\times$, $\mathfrak{b}$ is an ideal in K_0 divisible only by primes in $S_{K_0}^{\mathrm{ns}}$, and $\mathfrak{c}$ is the lift of an ideal from k_0 to K_0. It follows that $\mathfrak{a}^{p(1-\tau)} = (\gamma_{\mathfrak{a}}^{1-\tau})$.

Claim. *If $\iota(\bar{\mathfrak{a}})$ is trivial in A_1, then after multiplying $\gamma_{\mathfrak{a}}$ by a p-power root of unity if necessary, we have $K_1 = K_0(\sqrt[p]{\gamma_{\mathfrak{a}}^{1-\tau}})$.*

Proof of Claim. Let $\mathcal{O}_1$ be the ring of integers in K_1. From our assumption, $\mathfrak{a}^{1-\tau}\mathcal{O}_1$ is principal, generated by some anti-unit γ. The elements γ^p and $\gamma_{\mathfrak{a}}^{1-\tau}$ are both anti-unit generators the same principal ideal in K_1, so differ by a factor of a root of unity. When K_1 and K_0 contain the same number of p-power roots of unity (possibly 0), then by multiplying γ by a root of unity of order relatively prime to p and multiplying $\gamma_{\mathfrak{a}}$ by a p-power root of unity, we may ensure that

$$\mathfrak{a}^{1-\tau}\mathcal{O}_1 = \left(\sqrt[p]{\gamma_{\mathfrak{a}}^{1-\tau}}\right).$$

As $\mathfrak{a}^{1-\tau}$ is not principal in K_0, the above pth root generates K_1 over K_0. Otherwise, K_1 contains p-power roots of unity not contained in K_0. Since $[K_1 : K_0] = p$, we can write $K_1 = K_0(\zeta_{p^r})$ for some r. In this case, we can still adjust γ by a root of unity of order relatively prime to p so that $\gamma^p\zeta_{p^r} = \gamma_{\mathfrak{a}}^{1-\tau}$ for some p^rth root of unity ζ_{p^r}. Then $K_0(\sqrt[p]{\gamma_{\mathfrak{a}}^{1-\tau}})$ is a field intermediate between K_0 and $K_0(\zeta_{p^{r+1}})$, of degree p over K_0. It follows that in this case as well, $\sqrt[p]{\gamma_{\mathfrak{a}}^{1-\tau}}$ generates K_1 over K_0. The claim is proved. □

Cases $A^\sharp$ and $\flat$. Assume that $\iota(\bar{\mathfrak{a}})$ is trivial for some class $\bar{\mathfrak{a}}$ of order p in $\mathfrak{C}_0$. If K_1/k_0 is in case $\flat$, then K_0 does not contain the pth roots of unity. It follows from the claim that K_1/K_0 is not Galois, a contradiction. If K_1/k_0 is in case $A^\sharp$, then K_1/K_0 is generated by a p-power root of unity. Kummer theory shows that $\gamma_{\mathfrak{a}}^{1-\tau}$ differs from a pth power by a root of unity. Thus, $\mathfrak{a}^{p(1-\tau)} = (\gamma_{\mathfrak{a}}^{1-\tau})$ is the pth power of a principal ideal, contradicting the assumption that $\bar{\mathfrak{a}}$ has order p in A_0. Therefore, in either case $\flat$ or case $A^\sharp$, the restriction of ι,

$$\iota : {}_p\mathfrak{C}_0 \to A_1,$$

is an injection, which proves Eq. (2b).

Case $B^\sharp$. Assume that $\mathfrak{b}$ is a second fractional ideal of K_0 representing a class of order p in A_0 and such that $\iota(\bar{\mathfrak{b}})$ represents the trivial class in A_1. As above, there exists an element $\gamma_{\mathfrak{b}}$ in $K_0^\times$ such that $\mathfrak{b}^{p(1-\tau)} = (\gamma_{\mathfrak{b}}^{1-\tau})$ and such that $\sqrt[p]{\gamma_{\mathfrak{b}}^{1-\tau}}$ generates K_1 over K_0. Kummer theory then shows that there exists an integer e and an element β in $K_0^\times$ such that $\gamma_{\mathfrak{a}}^{1-\tau} = \beta^p\gamma_{\mathfrak{b}}^{e(1-\tau)}$. The following ideals in K_0 are thus equal: $\mathfrak{a}^{1-\tau} = (\beta)\mathfrak{b}^{e(1-\tau)}$. It follows that $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}^e$ as classes in A_0. Therefore, in case $B^\sharp$, the subgroup of A_0 consisting of classes that become trivial in A_1 is cyclic of order 1 or p, and inequality (2a) follows. □

To study case $B^\sharp$, we begin with a lemma which is taken from the remark preceding Proposition 2.2 in [4]. As the proof is short, it is reproduced here.

Lemma 3.4. *If K_1/k_0 is in case $B^\sharp$, then K_1/K_0 is unramified away from p.*

Proof. Recall that in case $B^\sharp$, K_1 contains ζ_p. Since $[K_1 : K_0] = p$, ζ_p is in K_0 as well. Suppose that the prime ideal $\mathfrak{p}$ of K_0 does not divide p and ramifies in K_1/K_0. The prime $\mathfrak{p}^+$ below $\mathfrak{p}$ in k_0 is tamely ramified in k_1/k_0. Thus, the tame inertia group of $\mathfrak{p}^+$, which injects into the multiplicative group of the residue field of the prime $\mathfrak{P}^+$ in k_1 above $\mathfrak{p}^+$ [10, Ch. 4, §2, Corollary 1], has order p. The absolute norm of $\mathfrak{P}^+$, hence of $\mathfrak{p}^+$, must therefore be congruent to 1 modulo p. Then since $K_0 = k_0(\zeta_p)$, $\mathfrak{p}^+$ splits in K_0/k_0. This is impossible since K_1/k_0 is in case $B^\sharp$. □

Lemma 3.5. *If K_1/k_0 is in case $B^\sharp$, then the kernel of $(\iota : A_0 \to A_1)$ is nontrivial.*

Proof. Let $\gamma \in K_0$ be an element such that $K_1 = K_0(\sqrt[p]{\gamma})$. As K_1/k_0 is Abelian and $1+\tau \in \operatorname{Ann}_{\mathbb{Z}[G]}\mu_0$, there is an element $\alpha \in K_0^\times$ such that $\gamma^{1+\tau} = \alpha^p$. Thus, $\gamma^{1-\tau} = \alpha^{-p}\gamma^2$, and hence $K_1 = K_0(\sqrt[p]{\gamma^{1-\tau}})$.

Now by Lemma 3.4, K_1/k_0 is unramified away from p. Therefore, the factorization of (γ) in K_0 has the form

$$(\gamma) = \mathfrak{b}^p \prod \mathfrak{p}_i^{e_i},$$

where the ideals $\mathfrak{p}_i$ are the prime ideals of K_0 dividing p that ramify in K_1/K_0. Since K_1/k_0 is in case $B^\sharp$, none of the prime ideals of k_0 lying below the ideals $\mathfrak{p}_i$ split in K_0/k_0. Thus,

$$\left(\gamma^{1-\tau}\right) = \left(\mathfrak{b}^{1-\tau}\right)^p.$$

If $\mathfrak{b}^{1-\tau}$ were principal, say equal to (β), then we would have $\gamma^{1-\tau} = \beta^p u$ for some unit u. Applying $1-\tau$, we find that

$$\left(\gamma^{1-\tau}\right)^2 = \left(\beta^{1-\tau}\right)^p u^{1-\tau} = \left(\beta^{1-\tau}\right)^p \zeta$$

for some root of unity ζ. It follows that $K_1 = K_0(\sqrt[p]{\gamma^{1-\tau}}) = K_0(\sqrt[p]{\zeta})$. This is a contradiction, since an extension K_1/k_0 in case $B^\sharp$ is not generated by roots of unity.

Thus, the ideal $\mathfrak{b}^{1-\tau}$ represents a class of order p in A_0. However, the lift of $\mathfrak{b}^{1-\tau}$ to K_1 is principal, generated by $\sqrt[p]{\gamma^{1-\tau}}$. $\square$

By Lemma 3.5 and the proof of Lemma 3.3,

$$\operatorname{rk}_p\left(\iota({}_p\mathfrak{C}_0)\right) = \operatorname{rk}_p(\mathfrak{C}_0) - 1$$

in case $B^\sharp$. Thus, Lemma 3.2 indicates that $\operatorname{rk}_{(1-\zeta_p)}(\mathfrak{K})$ is either equal to $\operatorname{rk}_p(\mathfrak{C}_0) - 1$ or to $\operatorname{rk}_p(\mathfrak{C}_0)$.

In fact, both possibilities occur. For instance, when $k_0 = \mathbb{Q}(\sqrt{29})$, computations done with PARI/GP show that k_0 has three cubic extensions unramified away from 3 and Abelian over k_0 aside from the first layer of the cyclotomic $\mathbb{Z}_3$-extension. If k_1 is one of these extensions and $K_1 = k_1(\zeta_3)$, then K_1/k_0 is in case $B^\sharp$, $\operatorname{rk}_3(\mathfrak{C}_0) = 1$, and $\operatorname{rk}_{(1-\zeta_3)}(\mathfrak{K}) = 0$. When $k_0 = \mathbb{Q}(\sqrt{62})$, then there are again three extensions other than the first layer of the cyclotomic $\mathbb{Z}_3$-extension. Constructing K_1 as before once again gives an extension in case $B^\sharp$, but in this case $\operatorname{rk}_3(\mathfrak{C}_0) = \operatorname{rk}_{(1-\zeta_3)}(\mathfrak{K}) = 1$.

Lemma 3.6. *Assume that the extension K_1/k_0 is such that no prime ideal splits in K_0/k_0 and ramifies in k_1/k_0. Then $\mathrm{Cl}_{K_0}^-\{p\}$ is a cyclic group if and only if $\mathrm{Cl}_{K_1}^-\{p\}$ is a cyclic $\mathbb{Z}[G]$-module.*

Proof. We will prove the corresponding statement for the G-modules A_0 and A_1 isomorphic to $\mathrm{Cl}_{K_0}^-\{p\}$ and $\mathrm{Cl}_{K_1}^-\{p\}$ by Lemma 2.1.

If A_1 is a cyclic $\mathbb{Z}[G]$-module, then A_0 is a cyclic group, being a surjective image of A_1 (by Proposition 2.2) with trivial G-action.

Conversely, assume that A_0 is cyclic. By Lemma 3.3, $\operatorname{rk}_{(1-\zeta_p)}(\mathfrak{K}) = 0$ or 1. When this rank is 0, $A_1 \cong A_0$, so let us assume that it is 1. If $|A_0| = 1$, then $|A_1| = p$, so A_1 is cyclic. Thus, assume that A_0 is nontrivial.

Let $|A_1| = p^s$ and $|A_0| = p^t$, so that $|\mathfrak{K}_p| = p^{s-t}$. Let M be the subgroup of A_1 generated by $\iota(A_0)$ and $\mathfrak{K}_p$. We note that M is a $\mathbb{Z}[G]$-module. Since $\mathrm{N} \circ \iota(\bar{\mathfrak{a}}) = \bar{\mathfrak{a}}^p$ for classes $\bar{\mathfrak{a}}$ in A_0, the restriction of the norm map to M is a surjection onto A_0^p with kernel $\mathfrak{K}_p$. Thus, $|M| = p^{t-1} \cdot |\mathfrak{K}_p| = p^{s-1}$.

Let $\bar{\mathfrak{a}}$ be a class in $A_1 \setminus M$. We will show that $\bar{\mathfrak{a}}$ generates A_1 as a $\mathbb{Z}[G]$-module. Let $\bar{\mathfrak{b}}$ be another class in A_1. Since A_1/M is cyclic of order p, there exist classes $\bar{\mathfrak{c}}$ in A_0 and $\bar{\mathfrak{d}} \in \mathfrak{K}_p$ and an integer e_1 such that $\bar{\mathfrak{b}} = \bar{\mathfrak{a}}^{e_1}\iota(\bar{\mathfrak{c}})\bar{\mathfrak{d}}$.

Now $\mathrm{N}(M) = A_0^p$, and as A_0 is cyclic and nontrivial, A_0/A_0^p has order p. Thus, since the norm map from A_1 to A_0 is surjective, it induces an isomorphism

$$\mathrm{N} : A_1/M \to A_0/A_0^p.$$

Therefore, A_0 is generated by $\mathrm{N}(\bar{\mathfrak{a}})$, and $\iota(\bar{\mathfrak{c}}) = \bar{\mathfrak{a}}^{e_2 N_H}$ for some integer e_2.

Next, by assumption, $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}_p) = 1$. Multiplication by $1-\sigma$ maps M into $\mathfrak{K}_p^{1-\sigma}$. Lemma 2.3 then shows that multiplication by $1-\sigma$ induces an isomorphism

$$A_1/M \cong \mathfrak{K}_p/(1-\zeta_p)\mathfrak{K}_p.$$

It follows that $\bar{\mathfrak{a}}^{1-\sigma}$ generates $\mathfrak{K}_p$ as a $\mathbb{Z}[G]$-module, so there is an $\alpha \in \mathbb{Z}[G]$ for which $\bar{\mathfrak{d}} = \bar{\mathfrak{a}}^{\alpha}$.

Finally, assembling these results, we find that

$$\bar{\mathfrak{b}} = \bar{\mathfrak{a}}^{e_1 + e_2 N_H + \alpha},$$

which proves that $\bar{\mathfrak{a}}$ generates A_1 as a $\mathbb{Z}[G]$-module. □

4. The p-primary Brumer–Stark conjecture

We retain the notation of Section 2. We will analyze the p-primary Brumer–Stark conjecture for a degree $2p$ extension K_1/k_0 by considering each component θ_1 and θ_0 of θ separately. Let p^r be the exact number of p-power roots of unity in K_1. The conjecture would follow immediately if we could show that $W_1\theta_1$ and $W_1\theta_0$ are both BS_{p^r}-annihilators for the group of fractional ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$. Unfortunately, this is not always true, and in fact, $W_1\theta_1$ and $W_1\theta_0$ do not necessarily even have p-integral coefficients. However, when no prime ideal both splits in K_0/k_0 and ramifies in k_1/k_0 and either $\mathrm{Cl}^-_{K_1}\{p\}$ is not a cyclic $\mathbb{Z}[G]$-module or S contains an unramified prime ideal of inertia degree p, we can prove the p-primary Brumer–Stark conjecture using the above strategy. We note that when S contains an unramified prime ideal of inertia degree p, $W_1\theta_0$ is 0 because the prime splits in K_0/k_0.

Theorem 4.1. *Let K_1/k_0 be an Abelian degree $2p$ extension of number fields, and let S be a set of places of k_0 containing the Archimedean places and the prime ideals that ramify in K_1. Assume that K_1 contains the p^rth roots of unity and that no prime ideal both splits in K_0/k_0 and ramifies in k_1/k_0. If $\mathrm{Cl}^-_{K_1}\{p\}$ is not a cyclic $\mathbb{Z}[G]$-module, or equivalently, if $\mathrm{Cl}^-_{K_0}\{p\}$ is not a cyclic group, then $W_1\theta_0$ is a BS_{p^r}-annihilator for the fractional ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$.*

Proof. It suffices to prove the theorem when p^r is the exact number of p-power roots of unity in K_1.

Lemma 3.6 shows that the assumptions in this theorem on the p-primary class groups of K_0 and K_1 are equivalent. By Proposition 2.4,

$$W_1\theta_0 = q\mathrm{N}_H \frac{W_0}{p}\tilde{\theta}_0,$$

where $W_1 = qW_0$ and $\tilde{\theta}_0$ is an arbitrary lift of $\theta_{K_0/k_0,S}$ to $\mathbb{Q}[G]$. We may assume that no place in S splits in K_0/k_0, since otherwise $\theta_{K_0/k_0,S} = 0$ and the theorem is trivial. Since p divides $|\mathfrak{C}_0|$, Eq. (1) shows that $W_1\theta_0 \in \mathbb{Z}[G]$.

If K_1/k_0 is in case $B^\sharp$ or case $\flat$, then Lemma 3.1 shows that $N_H \frac{W_0}{p}\tilde{\theta}_0$, and hence $W_1\theta_0$, is a BS_{p^r}-annihilator for ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$. Otherwise, if K_1/k_0 is in case $A^\sharp$, then

Lemma 3.1 shows that $N_H \frac{W_0}{p}\tilde{\theta}_0$ is a $\mathrm{BS}_{p^{r-1}}$-annihilator for the fractional ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$. Therefore, since p divides q, $W_1\theta_0$ is a BS_{p^r}-annihilator for ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$. $\square$

Theorem 4.2. *Let K_1/k_0 be an Abelian degree $2p$ extension of number fields, and let S be a set of places of k_0 containing the Archimedean places and the prime ideals that ramify in K_1. Assume that K_1 contains the p^rth roots of unity and that no prime ideal both splits in K_0/k_0 and ramifies in k_1/k_0. If $\mathrm{Cl}^-_{K_1}\{p\}$ is not a cyclic $\mathbb{Z}[G]$-module, or equivalently, if $\mathrm{Cl}^-_{K_0}\{p\}$ is not a cyclic group, or else if S contains a prime ideal that is unramified in K_1/k_0 and has inertia degree p, then $W_1\theta_1$ is a BS_{p^r}-annihilator for the fractional ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$.*

Proof. The proof of Theorem 4.1 showed that under the hypotheses of the present theorem, $W_1\theta_0$ is in $\mathbb{Z}[G]$. The integrality property of θ then shows that $W_1\theta_1$ is also in $\mathbb{Z}[G]$. It suffices to prove that $W_1\theta_1$ is a BS_{p^r}-annihilator for fractional ideals representing classes in $\mathrm{Cl}^-_{K_1}\{p\}$ since $W_1\theta_1$ is in $(1-\tau)\mathbb{Z}[G]$. We also need only to prove the theorem when p^r is the exact number of p-power roots of unity in K_1.

Case $A^\sharp$. Suppose that K_1/k_0 is in case $A^\sharp$. We will first see that $W_1\theta_1$ annihilates $\mathrm{Cl}^-_{K_1}\{p\}$. Let $\mathfrak{f}$ be an ideal representing a class in this group. Let N_σ be an integer such that

$$\zeta^\sigma = \zeta^{N_\sigma}$$

for all ζ in μ_1. Set

$$c = \frac{(N_\sigma)^p - 1}{p^r}.$$

Since K_0 contains the p^{r-1}th, but not the p^rth, roots of unity, c is an integer relatively prime to p. Thus, we can find a cth root of the class of $\mathfrak{f}$ in $\mathrm{Cl}^-_{K_1}\{p\}$, i.e., we can find an ideal $\mathfrak{a}$ in K_1 representing a class in $\mathrm{Cl}^-_{K_1}\{p\}$ and an element η in $K_1^\times$ such that $\mathfrak{f}(\eta) = \mathfrak{a}^c$. By Lemma 2.1, $\mathfrak{a}$ represents a class in A_1. Therefore

$$\bar{\mathfrak{a}}^{1-\sigma} \in \mathfrak{K}_p.$$

By the integrality property of θ, $(N_\sigma - \sigma)\theta$ is in $\mathbb{Z}[G]$. As W_0 divides $N_\sigma - 1$ and p divides $|\mathfrak{C}_0|$, Proposition 2.4 and Eq. (1) in Section 2 show that $(N_\sigma - \sigma)\theta_0$ is in $\mathbb{Z}[G]$. Thus, $(N_\sigma - \sigma)\theta_1$ is in $\mathbb{Z}[G]$ as well. Lemma 2.8 then shows that we can write $(N_\sigma - \sigma)\theta_1 = (1-\sigma)\alpha(1-\tau)$ with α in $\mathbb{Z}[H]$.

Let p^t be the exact power of p dividing $|\mathfrak{K}|$. Let v be the normalized valuation on $\mathbb{Z}[\zeta_p]$ corresponding to the prime ideal $(1-\zeta_p)$. Proposition 2.5 shows that $v(\chi(\alpha)) \geqslant t-1$. If $\mathrm{Cl}^-_{K_0}\{p\}$ is not a cyclic group, so that $\mathrm{rk}_p(A_0) \geqslant 2$, then $\mathrm{rk}_{(1-\zeta_p)}\mathfrak{K} \geqslant 2$ by Lemma 3.3. It follows that α annihilates $\mathfrak{K}_p$. Otherwise, if S contains a prime ideal that is unramified in K_1/k_0 and has inertia degree p, then $|S_2| \geqslant 1$ and so $v(\chi(\alpha)) \geqslant t$. Thus, α annihilates $\mathfrak{K}_p$ in this case as well. In either case

$$\mathfrak{a}^{(1-\sigma)\alpha} = \mathfrak{b}\mathfrak{c}(\tilde{\gamma}),$$

where $\mathfrak{b}$ is an ideal in K_1 divisible only by primes in $S^{\mathrm{ns}}_{K_1}$, $\mathfrak{c}$ is the lift of an ideal from k_1 to K_1, and $\tilde{\gamma}$ is in $K_1^\times$. As τ fixes $\mathfrak{b}$ and $\mathfrak{c}$, we obtain

$$\mathfrak{a}^{(N_\sigma - \sigma)\theta_1} = \left(\tilde{\gamma}^{1-\tau}\right). \tag{3}$$

Now we return to the relation $\mathfrak{f}(\eta) = \mathfrak{a}^c$ from above. Set $W_1 = p^r e$ with $(e, p) = 1$. Applying $\beta = e\sum_{i=0}^{p-1} N_\sigma^i \sigma^{p-1-i}$ to both sides of Eq. (3), we find that

$$\mathfrak{f}^{W_1\theta_1}\big(\eta^{W_1\theta_1}\big) = \mathfrak{a}^{cW_1\theta_1} = \mathfrak{a}^{e((N_\sigma)^p-1)\theta_1} = \big(\tilde{\gamma}^{\beta(1-\tau)}\big). \tag{4}$$

Thus, $W_1\theta_1$ annihilates the class of $\mathfrak{f}$ in $\mathrm{Cl}_{K_1}^-\{p\}$.

Let $g = \sigma\tau$, so g generates G. Let $N_g = -N_\sigma$, so g acts on roots of unity by raising to the N_g power. We observe that for an element of G, say g^j, the (anti-unit) generator $\gamma = \tilde{\gamma}^{\beta(1-\tau)}\eta^{-W_1\theta_1}$ of the principal ideal $\mathfrak{f}^{W_1\theta_1}$ is such that $\gamma^{N_g^j - g^j} \in K_1^{\times p^r}$. It suffices to show this when $j = 1$. In this case, since $\gamma^{1+\tau} = 1$, the condition is equivalent to $\gamma^{N_\sigma - \sigma} \in K_1^{\times p^r}$. This holds since

$$\begin{aligned}\gamma^{N_\sigma-\sigma} &= \tilde{\gamma}^{\beta(1-\tau)(N_\sigma-\sigma)}\eta^{-W_1\theta_1(N_\sigma-\sigma)}\\ &= \tilde{\gamma}^{ecp^r(1-\tau)}\eta^{-p^r e\theta_1(N_\sigma-\sigma)}\end{aligned}$$

and $(N_\sigma - \sigma)\theta_1$ is in $\mathbb{Z}[G]$.

Letting ξ be a p^rth root of γ, it follows that $K_1(\xi)/k_0$ is a Galois extension, say with Galois group $\tilde{G}$. Let $\tilde{g}$ be a lift of g to $\tilde{G}$. For each j such that $0 \leqslant j \leqslant 2p-1$, we have shown that there exists $\tilde{\xi}_j \in K_1^\times$ such that

$$\xi^{(N_g)^j - \tilde{g}^j} = \tilde{\xi}_j.$$

If $h \in \mathrm{Gal}(K_1(\xi)/K_1)$ and ζ is the p^rth root of unity such that $\xi^h = \zeta\xi$, then

$$\begin{aligned}\xi^{h\tilde{g}^j - \tilde{g}^j h} &= \xi^{((N_g)^j - \tilde{g}^j)h}\xi^{h(\tilde{g}^j - (N_g)^j)}\\ &= \tilde{\xi}_j^h(\zeta\xi)^{(\tilde{g}^j-(N_g)^j)}\\ &= \tilde{\xi}_j\tilde{\xi}_j^{-1} = 1.\end{aligned}$$

It follows that $\tilde{G}$ is a central extension of G by $\mathrm{Gal}(K_1(\xi)/K_1)$, hence is Abelian since G is cyclic. Therefore, γ is p^r-Abelian for K_1/k_0.

Case $B^\sharp$**.** Now suppose that K_1/k_0 is in case $B^\sharp$. Using the integrality property of θ and Proposition 2.4, we find

$$(1-\sigma)q\theta_1 = (1-\sigma)q\theta \in \mathbb{Z}[G].$$

Lemma 2.8 then shows that

$$(1-\sigma)q\theta_1 = (1-\sigma)\alpha(1-\tau),$$

with α in $\mathbb{Z}[H]$.

By Proposition 2.5,

$$N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}\big(\big(\chi(\alpha)\big)\big) = q^{p-2}2^{(p-1)(|S_1|-1)}p^{|S_2|}|\mathfrak{K}|. \tag{5}$$

If $\mathrm{rk}_p(A_0) \geqslant 2$, then Lemma 3.3 shows that p divides $|\mathfrak{K}|$. Otherwise, if S contains a prime ideal that is unramified in K_1/k_0 and splits in K_0/k_0, then $|S_2| \geqslant 1$. In either case, there is a factorization $\chi(\alpha) = (1-\zeta_p)\chi(\beta)$ with β in $\mathbb{Z}[H]$. Then $\chi(\alpha) = \chi((1-\sigma)\beta)$, so that $(1-\sigma)\alpha = (1-\sigma)^2\beta$.

To show that $W_1\theta_1$ annihilates $\mathrm{Cl}^-_{K_1}\{p\}$, let $\mathfrak{a}$ be a fractional ideal representing a class in that group. We first consider the case where either $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) \geqslant 2$ or $|S_2| \geqslant 1$; Eq. (5) shows that β annihilates $\mathfrak{K}_p$. As in case $A^\sharp$, we have

$$\mathfrak{a}^{(1-\sigma)\beta} = \mathfrak{b}\mathfrak{c}(\eta)$$

where $\mathfrak{b}$ is an ideal in K_1 divisible only by primes in $S^{\mathrm{ns}}_{K_1}$, $\mathfrak{c}$ is a lift from k_1, and η is in $K_1^\times$. Since $\mathfrak{b}$ and $\mathfrak{c}$ are both fixed by τ, it follows that

$$\mathfrak{a}^{(1-\sigma)q\theta_1} = \mathfrak{a}^{(1-\sigma)^2\beta(1-\tau)} = \left(\eta^{(1-\sigma)(1-\tau)}\right).$$

Now assume that $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) < 2$. As $\mathrm{rk}_p(\mathfrak{C}_0) \geqslant 2$, Lemma 3.3 shows that $\mathrm{rk}_p(\mathfrak{C}_0) = 2$ and $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) = 1$. Let $\iota : \mathfrak{C}_0 \to \mathfrak{C}_1$ be the map induced by lifting fractional ideals. By Eq. (2a) and Lemma 3.5, $\mathrm{rk}_p(\iota({}_p\mathfrak{C}_0)) = 1$. It follows that

$$\iota({}_p\mathfrak{C}_0) = \mathfrak{K}_p^H.$$

Once again, $\bar{\mathfrak{a}}^{(1-\sigma)}$ is in $\mathfrak{K}_p$, and now $\bar{\mathfrak{a}}^{(1-\sigma)\beta}$ is in $\mathfrak{K}_p^H$. Therefore, there exist η in $K_1^\times$ and fractional ideals $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{f}$ such that $\mathfrak{b}$ is supported at primes in $S^{\mathrm{ns}}_{K_1}$, $\mathfrak{c}$ is a lift from k_1, and $\mathfrak{f}$ is in ${}_p\mathfrak{C}_0$, and such that $\mathfrak{a}^{(1-\sigma)\beta} = \mathfrak{b}\mathfrak{c}\mathfrak{f}(\eta)$. Then

$$\mathfrak{a}^{(1-\sigma)q\theta_1} = \mathfrak{a}^{(1-\sigma)^2\beta(1-\tau)} = \left(\eta^{(1-\sigma)(1-\tau)}\right).$$

In summary, regardless of the $(1-\zeta_p)$-rank of $\mathfrak{K}$, there exists an anti-unit $\tilde{\gamma}$ such that $\tilde{\gamma}^{N_H} = 1$ and

$$\mathfrak{a}^{(1-\sigma)q\theta_1} = (\tilde{\gamma}).$$

Next, $N_H\theta_1 = 0$, so $\prod_{i=1}^{p-1}(1-\sigma^i)\theta_1 = p\theta_1$ by Lemma 2.6. Since p does not divide q in case $B^\sharp$, there is an integer $\tilde{e}$ such that $W_1 = \tilde{e}qp^r$. Then $\mathfrak{a}^{W_1\theta_1} = (\gamma)$, where

$$\gamma = \tilde{\gamma}^{p^{r-1}\tilde{e}\prod_{i=2}^{p-1}(1-\sigma^i)}.$$

As in case $A^\sharp$, the condition that $K_1(\sqrt[p^r]{\gamma})/k_0$ is Abelian is equivalent to the statement that

$$\gamma^{N_\sigma - \sigma} \in K_1^{\times p^r}.$$

Since $N_\sigma \equiv 1 \pmod{p^r}$, Lemma 2.6 shows this inclusion is equivalent to

$$\gamma^{1-\sigma} = \tilde{\gamma}^{W_0 - p^{r-1}\tilde{e}N_H} \in K_1^{\times p^r}.$$

Finally, this inclusion holds since $\tilde{\gamma}^{N_H} = 1$.

Case $\flat$**.** Finally, assume that K_1/k_0 is in case $\flat$. Let $\mathfrak{a}$ be a fractional ideal of K_1 representing a class in $\mathrm{Cl}^-_{K_1}\{p\}$. Since $p \nmid W_1$, we must simply show that $\mathfrak{a}^{W_1\theta_1}$ is principal, generated by an anti-unit.

Write $W_1\theta_1 = (1-\sigma)\alpha(1-\tau)$, as in Lemma 2.8. If $\mathrm{Cl}^-_{K_0}\{p\}$ is not a cyclic group, then Lemma 3.3 shows that $\mathrm{rk}_{(1-\zeta_p)}(\mathfrak{K}) \geqslant 2$. Once again, in either this case or the one where $|S_2| \geqslant 1$, Proposition 2.5 shows that α annihilates $\mathfrak{K}_p$. The proof that $\mathfrak{a}^{W_1\theta_1}$ is principal, generated by an anti-unit, now follows by an argument similar to that used to prove annihilation in cases $A^\sharp$ and $B^\sharp$. $\square$

Since the BS_{p^r}-annihilators for the fractional ideals representing classes in $\mathrm{Cl}_{K_1}\{p\}$ form an ideal in $\mathbb{Z}[G]$, Theorems 4.1 and 4.2, along with Propositions 2.1 and 2.2 in [4], imply Theorem 1.3.

Remark. A similar argument shows that the p-primary Brumer's conjecture always holds in case $B^\sharp$. Unfortunately, in that case the p-power roots of unity in K_1 are not cohomologically trivial so that Brumer's conjecture is weaker than the Brumer–Stark conjecture (see [4] and [8]).

5. The Brumer–Stark conjecture for degree 2p extensions

We retain the notation of Section 2. In this section, we will examine the p'-primary Brumer–Stark conjecture for primes $p' \neq p$, allowing the proof of the full Brumer–Stark conjecture for some classes of extensions of degree $2p$.

Let $\mathrm{Fit}_{\mathcal{O}}\,\mathfrak{K}$ denote the (zeroth) Fitting ideal of the module $\mathfrak{K}$ over the Dedekind domain $\mathcal{O} = \mathbb{Z}[\zeta_p]$. Let $\mathfrak{N}$ be the absolute norm of nonzero ideals of $\mathcal{O}$. We have

$$|\mathfrak{K}| = \mathfrak{N}\big(\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})\big).$$

The quotient $\tilde{\mu} = \mu_1/\mu_0$ of the groups of roots of unity in K_1 and K_0 is annihilated by N_H, giving it an $\mathcal{O}$-module structure (like the $\mathcal{O}$-module structure of $\mathfrak{K}$, it depends on the choice of a character generating $\hat{G}$), and $q = \mathfrak{N}(\mathrm{Fit}_{\mathcal{O}}(\tilde{\mu}))$. We may therefore rewrite Proposition 2.5 as

$$\mathfrak{N}\big(\mathrm{Fit}_{\mathcal{O}}(\tilde{\mu})\chi(\theta_1)\big) = \mathfrak{N}\big(2^{|S_1|}(1-\zeta_p)^{|S_2|}\,\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})\big). \tag{6}$$

This raises the following question:

Question. Is there an equality of ideals:

$$\mathrm{Fit}_{\mathcal{O}}(\tilde{\mu})\chi(\theta_1) = 2^{|S_1|}(1-\zeta_p)^{|S_2|}\,\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})? \tag{7}$$

As the $\mathcal{O}$-module structures here depend on the choice of a character, while Eq. (6) is independent of this choice, we must be careful in specifying the $\mathcal{O}$-module structures in this question. The natural choice for the $\mathcal{O}$-module structures of both $\tilde{\mu}$ and $\mathfrak{K}$ is that provided by the character χ.

Proposition 5.1. *If p' is a prime number not equal to 2 or p and if the factors on both sides of Eq. (7) supported at primes dividing p' are equal, then the p'-primary Brumer–Stark conjecture for K_1/k_0 is true.*

Remark. A generalization of this result appears in [13]. When the p'-cyclotomic μ-invariant of K_1 is 0, the p'-primary Brumer–Stark conjecture for K_1/k_0 is true without other hypotheses by recent work of Greither and Popescu.

Proof of Proposition 5.1. Let $\mathfrak{g}$ be an ideal of K_1 representing a class in $\mathrm{Cl}_{K_1}\{p'\}$. Write $W_1 = p'^r e$ with $(e, p') = 1$. We must show that $W_1\theta$ is a $\mathrm{BS}_{p'^r}$-annihilator for $\mathfrak{g}$. As $p' \neq p$, we can find an ideal $\mathfrak{f}$ representing a class in $\mathrm{Cl}_{K_1}\{p'\}$ and an element ξ in $K_1^\times$ such that $\mathfrak{g} = \mathfrak{f}^p(\xi)$. Tate [15, section 2] showed that $W_1\theta$ is a BS-annihilator for the principal ideals in K_1. Thus, it suffices to show that $pW_1\theta$ is $\mathrm{BS}_{p'^r}$-annihilator for $\mathfrak{f}$. By Propositions 2.4 and 2.7 and the integrality property of $\theta_{K_0/k_0,S}$, the elements $pW_1\theta_0$ and $pW_1\theta_1$ are in $\mathbb{Z}[G]$. We will show that they are each $\mathrm{BS}_{p'^r}$-annihilators for $\mathfrak{f}$.

Proposition 2.4 shows that

$$\mathfrak{f}^{pW_1\theta_0} = (N_{K_1/K_0}\mathfrak{f})^{qW_0\theta_{K_0/k_0,S}}\mathcal{O}_{K_1}$$

where $\mathcal{O}_{K_1}$ is the ring of integers in K_1. By Lemma 3.1, the ideal on the right is principal, generated by the qth power of a W_0-Abelian (for K_0/k_0) anti-unit. It is therefore generated by a W_1-Abelian (for K_1/k_0) anti-unit. It follows that $pW_1\theta_0$ is a BS-annihilator for $\mathfrak{f}$.

We will now see that $pW_1\theta_1$ is a $\mathrm{BS}_{p'^r}$-annihilator for $\mathfrak{f}$. As before, let N_σ be an integer such that $\zeta^\sigma = \zeta^{N_\sigma}$ for all ζ in μ_1. Set $g = \sigma\tau$ and $N_g = -N_\sigma$. Then $N_g^p \equiv -1 \pmod{p'^r}$, and adding W_1 to N_g if necessary, we can ensure that

$$c = \frac{N_g^p + 1}{p'^r}$$

is an integer relatively prime to p'. We can thus find a cth root of the class of $\mathfrak{f}$ in $\mathrm{Cl}_{K_1}\{p'\}$, i.e., we can find an ideal $\mathfrak{a}$ in K_1 representing a class in $\mathrm{Cl}_{K_1}^-\{p'\}$ and an element η in $K_1^\times$ such that $\mathfrak{f}(\eta) = \mathfrak{a}^c$.

Next, observe that $(N_g - g)\theta$ is in $\mathbb{Z}[G]$ by the integrality property of θ, hence $(N_g - g)\frac{pW_1}{p'^r}\theta$ is as well. Moreover, the definition of θ_0 shows that it is in $N_H(1-\tau)\mathbb{Q}[G]$. Thus,

$$(N_g - g)\theta_0 = (N_g - \sigma\tau)\theta_0 = (N_g + 1)\theta_0. \tag{8}$$

As g acts on μ_0 as complex conjugation, W_0 divides $N_g + 1$. Eq. (1) and Proposition 2.4 then show that $(N_g - g)\theta_0$ is in $\frac{1}{p}\mathbb{Z}[G]$, and hence, $(N_g - g)\frac{pW_1}{p'^r}\theta_0$ is in $\mathbb{Z}[G]$. Since $\theta_1 = \theta - \theta_0$, $(N_g - g)\frac{pW_1}{p'^r}\theta_1$ is also in $\mathbb{Z}[G]$. By Lemma 2.8, we may write $(N_g - g)\frac{pW_1}{p'^r}\theta_1 = (1-\sigma)\alpha(1-\tau)$ with α in $\mathbb{Z}[H]$.

Let $\mathfrak{P}'$ be a prime ideal of $\mathbb{Z}[\zeta_p]$ dividing p', and let $v_{\mathfrak{P}'}$ be the normalized valuation corresponding to $\mathfrak{P}'$. By our assumption that the factors on both sides of Eq. (7) supported at primes dividing p' are equal (and that p' is not 2 or p), we have

$$\begin{aligned} v_{\mathfrak{P}'}\bigl(\chi(\alpha)\bigr) &= v_{\mathfrak{P}'}\bigl(N_g - \chi(g)\bigr) - v_{\mathfrak{P}'}\bigl(\mathrm{Fit}_{\mathcal{O}}(\tilde{\mu})\bigr) + v_{\mathfrak{P}'}\bigl(\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})\bigr) \\ &\geqslant v_{\mathfrak{P}'}\bigl(\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})\bigr), \end{aligned} \tag{9}$$

where the inequality holds because $N_g - \chi(g)$ is in $\mathrm{Ann}_{\mathcal{O}}(\tilde{\mu}) = \mathrm{Fit}_{\mathcal{O}}(\tilde{\mu})$.

Writing $\mathfrak{f}(\eta) = \mathfrak{a}^c$ as above, it then follows as in the derivation of Eq. (3) in the proof of Theorem 4.2 (replacing A_1 by the p'-primary part of $\mathfrak{C}_1$) that

$$\mathfrak{a}^{(N_g - g)\frac{pW_1}{p'^r}\theta_1} = \bigl(\tilde{\gamma}^{1-\tau}\bigr) \tag{10}$$

for some element $\tilde{\gamma}$ in $K_1^\times$. Applying $\beta = \sum_{i=0}^{p-1} N_g^i g^{p-1-i}$ to both sides of Eq. (10) and observing that θ_1 is in $(1-\tau)\mathbb{Q}[G]$ yields

$$\mathfrak{f}^{pW_1\theta_1}\bigl(\eta^{pW_1\theta_1}\bigr) = \mathfrak{a}^{cpW_1\theta_1} = \mathfrak{a}^{((N_g)^p + 1)\frac{pW_1}{p'^r}\theta_1} = \bigl(\tilde{\gamma}^{\beta(1-\tau)}\bigr).$$

Thus, $pW_1\theta_1$ annihilates the class of $\mathfrak{f}$.

We are now in essentially the same position as at Eq. (4) in the proof of case $A^\sharp$ of Theorem 4.2. The rest of the proof that $pW_1\theta_1$ is a $\mathrm{BS}_{p'^r}$-annihilator of $\mathfrak{f}$ follows as in that proof (with p replaced by p'). $\square$

The above proposition is too weak when $p' = 2$ since there is an additional factor $2^{|S_1|}$ appearing in Eq. (7). A similar extra power of 2 occurs in Tate's formula (1) and allowed him to prove a strengthened 2-primary Brumer–Stark conjecture for quadratic extensions. The following proposition is the analogous strengthening for degree $2p$ extensions. It was obtained independently by Greither, Roblot, and Tangedal in an unpublished manuscript [5].

Proposition 5.2. *Suppose that the number of* 2*-power roots of unity in* K_1 *is* 2^r*. If the factors of both sides of Eq.* (7) *supported at primes dividing* 2 *are equal, then* $\frac{1}{2^{|S_1|-2}} W_1\theta$ *is a* BS_{2^r}*-annihilator for the fractional ideals representing classes in* $\mathrm{Cl}_{K_1}\{2\}$*. If either* $S \neq S_1$ *or* $\mathfrak{C}_0 \otimes \mathbb{Z}_2$ *is not cyclic, then* $\frac{1}{2^{|S_1|-1}} W_1\theta$ *is a* BS_{2^r}*-annihilator for the fractional ideals representing classes in* $\mathrm{Cl}_{K_1}\{2\}$.

Proof. We mimic the proof of Proposition 5.1. First, we show that $\frac{1}{2^{|S_1|-1-\delta}} pW_1\theta_0$ is a BS_{2^r}-annihilator for ideals representing classes in $\mathrm{Cl}_{K_1}\{2\}$, where δ is 1 if $S = S_1$ and $\mathfrak{C}_0 \otimes \mathbb{Z}_2$ is cyclic and $\delta = 0$ otherwise. We then show that $\frac{1}{2^{|S_1|-1}} pW_1\theta_1$ is always a BS_{2^r}-annihilator for ideals representing classes in $\mathrm{Cl}_{K_1}\{2\}$. By reasoning similar to that at the beginning of the proof of Proposition 5.1, this will complete the proof.

Let $\mathfrak{f}$ be an ideal representing a class in $\mathrm{Cl}_{K_1}\{2\}$. Lemma 3.1 shows that $\frac{1}{2^{|S|-2}} W_0\theta_{K_0/k_0,S}$ is a BS-annihilator for the nonzero fractional ideals in K_0. Moreover, if $S \neq S_1$, then it follows immediately that in fact $\frac{1}{2^{|S_1|-1}} pW_0\theta_{K_0/k_0,S}$ is a BS-annihilator for the nonzero fractional ideals of K_0. If $\mathfrak{C}_0 \otimes \mathbb{Z}_2$ is not cyclic, then Lemma 3.1 gives the same result.

We show that $\frac{1}{2^{|S_1|-1}} pW_1\theta_1$ is a BS_{2^r}-annihilator for $\mathfrak{f}$ by modifying the analysis of $pW_1\theta_1$ in the proof of Proposition 5.1. Let $W_1 = 2^r e$ with e odd. We first show that $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_1$ is in $\mathbb{Z}[G]$. Let $\mathfrak{P}'$ be a prime ideal of $\mathbb{Z}[\zeta_p]$ dividing 2, and let $v_{\mathfrak{P}'}$ be the corresponding normalized valuation. Using Lemma 2.8, write $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_1 = (1-\sigma)\alpha(1-\tau)$ with α in $\mathbb{Q}[H]$. By the assumption that the factors of both sides of Eq. (7) supported at primes dividing 2 are equal (and observing that q is odd), we have

$$v_{\mathfrak{P}'}\big(\chi(\alpha)\big) = v_{\mathfrak{P}'}\big(N_g - \chi(g)\big) + v_{\mathfrak{P}'}\big(\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})\big). \tag{11}$$

Thus, since $\mathbb{Z}[\zeta_p]$ has an integral basis consisting of powers of ζ_p, we can write $\chi(\alpha)$ as a linear combination on this basis with 2-integral coefficients. Pulling this linear combination back through χ gives an element β in $\mathbb{Q}[H]$ with 2-integral coefficients that differs from α by a multiple of N_H. Then $(1-\sigma)\beta$, and hence $(1-\sigma)\alpha$, has 2-integral coefficients, and so $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_1$ does too. The integrality property of θ shows that $(N_g - g)pW_1\theta$ is in $\mathbb{Z}[G]$. Eq. (8) and formula (1) show that $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_0$ is in $\mathbb{Z}[G]$. It follows that $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_1$ is in $\mathbb{Z}[G]$ as well.

The proof that $(N_g - g)\frac{pW_1}{2^r 2^{|S_1|-1}}\theta_1$ annihilates $\mathrm{Cl}_{K_1}\{2\}$ follows using equality (11) above. The proof mimics the reasoning after inequality (9). The proof that $\frac{1}{2^{|S_1|-1}} pW_1\theta_1$ is a BS_{2^r}-annihilator for $\mathfrak{f}$ then follows from an argument similar to the end of the proof of Proposition 5.1. $\square$

The following theorem follows immediately from the above two propositions and Eq. (6).

Theorem 5.3. *If the prime number* p' *is inert in* $\mathbb{Z}[\zeta_p]$*, then the* p'*-primary Brumer–Stark conjecture for* K_1/k_0 *is true. If* 2 *is inert and the number of* 2*-power roots of unity in* K_1 *is* 2^r*, then* $\frac{1}{2^{|S_1|-2}} W_1\theta$ *is a* BS_{2^r}*-annihilator for the group of fractional ideals representing classes in* $\mathrm{Cl}_{K_1}\{2\}$*. If* 2 *is inert and either* $S \neq S_1$ *or* $\mathfrak{C}_0 \otimes \mathbb{Z}_2$ *is not cyclic, then* $\frac{1}{2^{|S_1|-1}} W_1\theta$ *is a* BS_{2^r}*-annihilator for the group of fractional ideals representing classes in* $\mathrm{Cl}_{K_1}\{2\}$.

Combining Theorems 1.3 and 5.3 with Propositions 1.3, 2.1, and 2.2 in [4], we immediately obtain the following result concerning the global Brumer–Stark conjecture (Proposition 1.3 in [4] actually allows for a somewhat more general statement).

Theorem 5.4. *Let* p *be an odd prime number having* 2 *as a primitive root. Suppose that* K_1/k_0 *is an Abelian degree* $2p$ *extension of number fields with* k_0 *a cyclic extension of* $\mathbb{Q}$ *and that* $\mathrm{Cl}_{K_1}^{-}\{p\}$ *is not a cyclic* $\mathbb{Z}[G]$*-module. Then the Brumer–Stark conjecture for* K_1/k_0 *is true.*

The analytic class number formula only gives information about the product of L-values, i.e., about their absolute norm, so the more refined algebraic information contained in formula (7) seems out of reach of this fundamental tool. However, the existence of a subfield of k_0 satisfying certain conditions puts a restriction on the Fitting ideals strong enough to verify this formula using only the class number formula. Tate [15, section 3, (e)] proved the Brumer–Stark conjecture for a class of cyclic quartic extensions satisfying a similar restriction.

Proposition 5.5. *The equality of ideals* (7) *is valid for extensions K_1/k_0 such that there exists another field k with $k \subset k_0 \subset K_1$ where K_1/k and k_0/k are Galois, the map* $\mathrm{Gal}(k_0/k) \to \mathrm{Aut}(\mathrm{Gal}(K_1/k_0))$ *given by the lift-and-conjugate action is surjective, and S is invariant under* $\mathrm{Gal}(k_0/k)$ (*in particular, when S is the minimal allowed set in the Brumer–Stark conjecture*).

Proof. Notice first that the action of $\mathrm{Gal}(k_0/k)$ on $\mathbb{Z}[G]$ induces a canonical map

$$\phi : \mathrm{Gal}(k_0/k) \to \mathrm{Gal}\big(\mathbb{Q}(\zeta_p)/\mathbb{Q}\big).$$

The surjectivity hypothesis in the proposition implies that ϕ is surjective.

We will now see that the $\mathcal{O}$-module $\tilde{\mu}$ is trivial. Let ρ be an element in $\mathrm{Gal}(k_0/k)$ that acts nontrivially on $\mathrm{Gal}(K_1/k_0)$, and let $\tilde{\rho}$ be a lift of ρ to $\mathrm{Gal}(K_1/k)$. Let c be an integer such that $\rho \cdot \sigma = \sigma^c$. Then $c \not\equiv 0$ or 1 (mod p). Let $\mathrm{N}_{\tilde{\rho}}$ be an integer such that $\zeta^{\tilde{\rho}} = \zeta^{\mathrm{N}_{\tilde{\rho}}}$ for all ζ in μ_1. Define N_σ similarly. Then letting $N_{\tilde{\rho}}^{-1}$ be an inverse for $N_{\tilde{\rho}}$ modulo W_1,

$$\zeta^{\mathrm{N}_\sigma^c} = \zeta^{\sigma^c} = \zeta^{\tilde{\rho}\sigma\tilde{\rho}^{-1}} = \zeta^{\mathrm{N}_{\tilde{\rho}}\mathrm{N}_\sigma \mathrm{N}_{\tilde{\rho}}^{-1}} = \zeta^{\mathrm{N}_\sigma}$$

for all ζ in μ_1. Therefore, $\mathrm{N}_\sigma^{c-1} \equiv 1 \pmod{W_1}$. Also, since $\sigma^p = 1$, we have $\mathrm{N}_\sigma^p \equiv 1 \pmod{W_1}$. But $(p, c-1) = 1$ so that $\mathrm{N}_\sigma \equiv 1 \pmod{W_1}$, and thus σ fixes μ_1. In other words, $\tilde{\mu}$ is trivial.

Next, one can show that $\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})$ and $\chi(\theta_1)$ are fixed by $\phi(\mathrm{Gal}(k_0/k))$. The first statement is a general fact about Fitting ideals of G-modules and can be proved directly from the definition. The second follows from the following observation: If $k \subset k_0 \subset K_1$ is a tower of number fields with K_1/k_0 Abelian and K_1/k and k_0/k Galois, S is a set of places of k_0 invariant under G, and if σ_1 and σ_2 are in the same orbit of $\mathrm{Gal}(K_1/k_0)$ under the lift and conjugate action, then there is an equality of functions $\zeta_{K_1/k_0,S}(s,\sigma_1) = \zeta_{K_1/k_0,S}(s,\sigma_2)$. This can be shown for $\Re(s) > 1$ from the series defining the functions, and hence the equality holds for all s.

Since ϕ is surjective, $\chi(\theta_1)$ is in $\mathbb{Q}$ and $\mathrm{Fit}_{\mathcal{O}}(\mathfrak{K})$ is the product of a lift from $\mathbb{Q}$ with a power of $(1-\zeta_p)$. The proposition now follows from Eq. (6). □

Remark. One might wonder if the field

$$\mathbb{Q}\big(\chi(\theta_1)\big) = \mathbb{Q}\big(L_{K_1/k_0,S}(0,\overline{\chi})\big)$$

is *characterized* by the fields k as in the proposition. More precisely, let T be the set of fields k where $k \subseteq k_0 \subseteq K_1$ with K_1/k and k_0/k both Galois. For each k in T, we have a function $\phi_k : \mathrm{Gal}(k_0/k) \to \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ induced by the action of $\mathrm{Gal}(k_0/k)$ on $\mathbb{Z}[G]$. Let H be the subset of $\mathrm{Gal}(K_1/k_0)$ generated by the images of the maps ϕ_k for k in T. The above proof demonstrates the containment

$$\mathbb{Q}\big(L_{K_1/k_0,S}(0,\overline{\chi})\big) \subseteq \mathbb{Q}(\zeta_p)^H.$$

It would be interesting to know if one always has

$$\mathbb{Q}\big(L_{K_1/k_0,S}(0,\overline{\chi})\big) = \mathbb{Q}(\zeta_p)^H.$$

Combining Theorem 1.3, Propositions 5.1, 5.2, 5.5, and Propositions 2.1 and 2.2 in [4], we obtain the following theorem:

Theorem 5.6. *Suppose that K_1/k_0 satisfies the assumptions of Proposition* 5.5 *and that* $\mathrm{Cl}_{K_1}^-\{p\}$ *is not a cyclic* $\mathbb{Z}[G]$*-module. Then the Brumer–Stark conjecture for K_1/k_0 is true.*

In [15, section 3, (d)], Tate explains the existence of his strengthening of the 2-primary Brumer–Stark conjecture for quadratic extensions. Inspired by lines 24–34 on page 199 of [14], he observes that his strong 2-primary conjecture for quadratic extensions implies the Brumer–Stark conjecture for certain larger Abelian extensions. By a similar argument, we can show that the strong form of the Brumer–Stark conjecture for a degree $2p$ extension K_1/k_0 suggested by Proposition 5.2 and Theorem 5.3 implies the standard Brumer–Stark conjecture for certain larger extensions.

The setup is as follows: Let K/k_0 be an Abelian extension of number fields with K totally complex and k_0 totally real. Let S be a set of places of k_0 containing the Archimedean places and the prime ideals that ramify in K/k_0. Let S' be a subset of S consisting of places whose decomposition groups have order 2. For each place ν in S', let τ_ν be the nontrivial element of the decomposition group of ν. Let $K_1 \subset K$ be the maximal subfield fixed by all products $\tau_\nu \tau_{\nu'}$ with ν and ν' in S'.

Proposition 5.7. *Let $k_0 \subset K_1 \subset K$ be as above. Assume that K_1/k_0 has degree $2p$ and that the following strengthening of the Brumer–Stark conjecture for K_1/k_0 holds: $\frac{1}{2^{|S_1|-2}} W_1\theta$ is a* BS*-annihilator for the nonzero fractional ideals in K_1, and if $S \neq S_1$, then $\frac{1}{2^{|S_1|-1}} W_1\theta$ is a* BS*-annihilator for the nonzero fractional ideals in K_1. Then the Brumer–Stark conjecture for K/k_0 holds.*

Proof. Let $\tilde{H} = \mathrm{Gal}(K/K_1)$. The proof of part (d) in section 3 of [15] shows that

$$\theta_{K/k_0,S} = \frac{N_{\tilde{H}}}{|\tilde{H}|}\tilde{\theta}_{K_1/k_0,S},$$

where $N_{\tilde{H}}$ is the algebraic norm element corresponding to $\tilde{H}$ and $\tilde{\theta}_{K_1/k_0,S}$ is a lift of $\theta_{K_1/k_0,S}$ to $\mathbb{Q}[\mathrm{Gal}(K/k_0)]$. We will show that $\frac{1}{|\tilde{H}|} W_1\theta_{K_1/k_0,S}$ is a BS-annihilator for the nonzero fractional ideals in K_1. It then follows immediately from the above formula that the Brumer–Stark conjecture for K/k_0 holds.

By our hypotheses, we only need to show that $|\tilde{H}| \leqslant 2^{|S_1|-1}$ unless $S = S_1$, in which case we must show that $|\tilde{H}| \leqslant 2^{|S_1|-2}$. The inequality $|\tilde{H}| \leqslant 2^{|S'|-1}$ holds since $\mathrm{Gal}(K/k_1)$ is generated by $|S'|$ elements τ_ν of order 2. Recall that S_1 is the set of places of S that split completely in k_1/k_0. As the decomposition group in $\mathrm{Gal}(K/k_0)$ of each place in S' has order 2 and k_1/k_0 has degree p, it follows that $S' \subseteq S_1$. Hence, $|\tilde{H}| \leqslant 2^{|S_1|-1}$, and in fact $|\tilde{H}| \leqslant 2^{|S_1|-2}$ if $S' \neq S_1$. Now assume that $S' = S_1 = S$. In order to show that $|\tilde{H}| \leqslant 2^{|S_1|-2}$, we must find a nontrivial relation between the automorphisms τ_ν. This exceptional case is similar to the one that arose in [15, section 3, (d)], and the relation can be found with the following modification of Tate's argument: Let $\tilde{G}$ be the Galois group of K/k_0. Let $-1 = (\ldots, -1_\nu, \ldots)$ be the idéle -1 of k_0. For each place ν of k_0, let $r_\nu : (k_0)_\nu^\times \to \tilde{G}_\nu$ be the local Artin map. We obtain $\prod_\nu r_\nu(-1_\nu) = 1$ by the reciprocity law. For places ν not in S, $r_\nu(-1_\nu) = 1$ since S contains the ramified places. Thus, $\prod_{\nu \in S} r_\nu(-1_\nu) = 1$. For each ν in $S = S'$, we have $r_\nu(-1_\nu) = 1$ or τ_ν, but for Archimedean ν, $r_\nu(-1_\nu) = \tau_\nu$ because $-1_\nu < 0$ is not a norm of $K_\nu \cong \mathbb{C}$. Thus, there is a nontrivial relation between the τ_ν. $\square$

Choosing S' to be the set of Archimedean places of k_0 shows that the strong Brumer–Stark conjecture for K_1/k_0 implies the standard Brumer–Stark conjecture for every extension containing K_1/k_0 as its maximal CM subextension.

Acknowledgments

I am indebted to Cristian Popescu for introducing me to this subject. I would also like to thank both Cristian Popescu and Karl Rubin for their guidance while this work was conducted and for helpful comments regarding the initial manuscript. Finally, I would like to thank the referee for quickly creating a detailed and helpful list of improvements to the paper.

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